Comment on "Dirac Quantization of Pais-Uhlenbeck Fourth Order Oscillator"

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Abstract

The structure of Pais-Uhlenbeck oscillator in the equal-frequency limit has been recently studied by Mannheim and Davidson [Phys.Rev. A71 (2005), 042110]. It appears that taking this limit , as presented in the above paper, is quite subtle and the resulting structure of space of states - involved. In order to clarify the situation we present here the proper way of taking the equal-frequency limit, first under the assumption that the scalar product in the space of states is positive defined. We discuss also the case of indefinite metric space of states. We show that, irrespective of the way the limit is defined, the limiting theory can be hardly viewed as satisfactory.

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Pais-Uhlenbeck quartic oscillator [1] is described by the Lagrangian

$$L = \frac{m}{2}\dot{q}^2 - \frac{m\omega^2}{2}q^2 - \frac{m\lambda}{2}\ddot{q}^2 \tag{1}$$

Its behaviour depends on actual values of parameters ω and λ. In what follows we keep m and ω fixed while varying λ. The relevant equation of motion reads

$$\lambda\left(\frac{d^2}{dt^2} + \omega_1^2\right)\left(\frac{d^2}{dt^2} + \omega_2^2\right)q(t) = 0 \tag{2}$$

with

$$\omega_{1,2}^2 \equiv \frac{1 \pm \sqrt{1 - 4\lambda\omega^2}}{2\lambda} \tag{3}$$

For large λ $\quad (\lambda > \frac{1}{4\omega^2})$ both frequencies are complex. On the other hand, in the range $0 < \lambda < \frac{1}{4\omega^2}$ they are real; in the limiting case $\lambda = \frac{1}{4\omega^2}$ there is a double degeneracy $\omega_1 = \omega_2 = \sqrt{2}\omega$. Finally, if $\lambda < 0$, one frequency is real while the second one - purely imaginary.

In order to quantize our theory one has to put it first in Hamiltonian form. This can be achieved within Ostrogradski formalism [2], [3]. It is well known [4], [5], [6] that the Ostrogradski procedure is essentially a form of Dirac method for constrained theories.
In our case one finds [7], [8] $\div$ [10] the following canonical variables

$$q_1 \equiv q, \quad q_2 \equiv \dot{q}$$

$$\begin{aligned}
\Pi_1 &\equiv \frac{\delta L}{\delta \dot{q}} = \frac{\partial L}{\partial \dot{q}} - \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{q}}\right) = m(\dot{q} + \lambda\,\dddot{q}) \\
\Pi_2 &\equiv \frac{\delta L}{\delta \ddot{q}} = \frac{\partial L}{\partial \ddot{q}} = -m\lambda\ddot{q}
\end{aligned} \tag{4}$$

together with the Hamiltonian

$$H = \Pi_1 q_2 - \frac{1}{2m\lambda}\Pi_2^2 + \frac{m\omega^2}{2}q_1^2 - \frac{m}{2}q_2^2 \tag{5}$$

Quantization can be now performed in the standard way by imposing the commutation rule

$$[\hat{q}_i, \hat{\Pi}_j] = i\hbar\delta_{ij} \tag{6}$$

Consider first the range $0 < \lambda < \frac{1}{4\omega^2}$; then $\omega_1^2 > \omega_2^2 > 0$. To make the structure of the Hamiltonian transparent we perform the following canonical transformation (cf.

Ref.[1])

$$
\begin{aligned}
\hat{q}_1 &= \frac{1}{\sqrt{\lambda(\omega_1^2-\omega_2^2)}}(-\hat{x}_1+\hat{x}_2) \\
\hat{q}_2 &= \frac{1}{m\sqrt{\lambda(\omega_1^2-\omega_2^2)}}(\hat{p}_1+\hat{p}_2) \\
\hat{\Pi}_1 &= \sqrt{\frac{\lambda}{\omega_1^2-\omega_2^2}}(\omega_2^2\hat{p}_1+\omega_1^2\hat{p}_2) \\
\hat{\Pi}_2 &= m\sqrt{\frac{\lambda}{\omega_1^2-\omega_2^2}}(-\omega_1^2\hat{x}_1+\omega_2^2\hat{x}_2)
\end{aligned}
\tag{7}
$$

Note that the above transformation becomes singular in the doubly degenerate limit $\lambda \to \frac{1}{4\omega^2}$. In term of new variables the Hamiltonian takes particularly simple form

$$
\hat{H} = \left(\frac{\hat{p}_2^2}{2m}+\frac{m\omega_2^2}{2}\hat{x}_2^2\right) - \left(\frac{\hat{p}_1^2}{2m}+\frac{m\omega_1^2}{2}\hat{x}_1^2\right) \tag{8}
$$

The eigenvectors of $\hat{H}$ are uniquely determined (up to a phase factor) by two nonnegative integers n_1, n_2

$$
\hat{H} \mid n_1, n_2\rangle = \left(-\hbar\omega_1(n_1+\frac{1}{2})+\hbar\omega_2(n_2+\frac{1}{2})\right) \mid n_1, n_2\rangle \tag{9}
$$

The spectrum of $\hat{H}$ is simple provided $\frac{\omega_1}{\omega_2}$ is irrational; for rational $\frac{\omega_1}{\omega_2}$ (superintegrable case) there is a degeneracy.
The wave functions in the coordinate representation read

$$
\begin{aligned}
&\langle x_1, x_2 \mid n_1, n_2\rangle = \\
&= N(n_1)N(n_2)\sqrt[4]{\frac{m^2\omega_1\omega_2}{\hbar^2}}H_{n_1}\left(x_1\sqrt{\frac{m\omega_1}{\hbar}}\right)H_{n_2}\left(x_2\sqrt{\frac{m\omega_1}{\hbar}}\right)\cdot e^{-\frac{m}{2\hbar}(\omega_1 x_1^2+\omega_2 x_2^2)}
\end{aligned}
\tag{10}
$$

where $N(n) \equiv (\sqrt{\pi}\cdot 2^n \cdot n!)^{-\frac{1}{2}}$.
The spectrum of $\hat{H}$, as given by eq.(9), is unbounded from below. One gets positive energy spectrum by admitting indefinite metric in the space of states. To this end we consider the space of states endowed with the positive-definite scalar product $(\cdot,\cdot)$ and define the "physical" scalar product with the help of metric operator η

$$
\langle \Phi \mid \Psi\rangle \equiv (\Phi, \eta\Psi), \quad \eta = \eta^+ = \eta^{-1} \tag{11}
$$

Denoting by "$\star$" the hermitean conjugate with respect to the scalar product $\langle\cdot\mid\cdot\rangle$ one finds for any operator $\hat{A}$.

$$
\hat{A}^\star = \eta\hat{A}^+\eta \tag{12}
$$

Let a_i, a_i^+ be the creation/anihilation operators constructed out of $\hat{x}_i, \hat{p}_i$,

$$\begin{aligned}\hat{x}_i &= i\sqrt{\frac{\hbar}{2m\omega_i}}\left(a_i - a_i^+\right)\\ \hat{p}_i &= \sqrt{\frac{m\hbar\omega_i}{2}}\left(a_i + a_i^+\right)\end{aligned}\tag{13}$$

We define

$$\eta = (-1)^{N_1} = e^{i\pi a_1^+ a_1}\tag{14}$$

Then

$$\langle n_1, n_2 \mid n_1', n_2'\rangle = (-1)^{n_1}\delta_{n_1 n_1'}\delta_{n_2 n_2'};\tag{15}$$

moreover, $\hat{x}_i^\star = \hat{x}_i,\ \hat{p}_i^\star = \hat{p}_i$ imply

$$\hat{x}_i^+ = (-1)^i\hat{x}_i, \quad \hat{p}_i^+ = (-1)^i\hat{p}_i\tag{16}$$

Defining (c.f. [1])

$$\begin{aligned}\hat{x}_1' &= \pm i\hat{x}_1, \quad \hat{x}_2' = \hat{x}_2\\ \hat{p}_1' &= \mp i\hat{p}_1, \quad \hat{p}_2' = \hat{p}_2\end{aligned}\tag{17}$$

we find that $\hat{x}_i', \hat{p}_i'$ are hermitean (with respect to "+" conjugation) and

$$\hat{H} = \left(\frac{\hat{p}_1'^2}{2m} + \frac{m\omega_1^2}{2}\hat{x}_1'^2\right) + \left(\frac{\hat{p}_2'^2}{2m} + \frac{m\omega_2^2}{2}\hat{x}_2'^2\right)\tag{18}$$

Therefore, the spectrum of $\hat{H}$ is now positive definite

$$E_{n_1,n_2} = \hbar\omega_1(n_1 + \frac{1}{2}) + \hbar\omega_2(n_2 + \frac{1}{2})\tag{19}$$

The "physical" subspace is spanned by the vectors $\mid 2n_1, n_2\rangle$.

Let us now consider the degenerate case $\lambda = \frac{1}{4\omega^2}$. To reveal the structure of the Hamiltonian we define new variables $\hat{Q}_i, \hat{P}_i$ by [1]

$$\begin{aligned}\hat{q}_1 &= \frac{\hat{Q}_1}{2\sqrt{2}} + \frac{\hat{P}_2}{m\omega}\\ \hat{q}_2 &= \frac{\omega\hat{Q}_2}{2} + \frac{\sqrt{2}\hat{P}_1}{m}\\ \hat{\Pi}_1 &= \frac{\hat{P}_1}{\sqrt{2}} - \frac{3}{4}m\omega\hat{Q}_2\\ \hat{\Pi}_2 &= \frac{\hat{P}_2}{2\omega} - \frac{3}{4\sqrt{2}}m\hat{Q}_1\end{aligned}\tag{20}$$

Eq.(20) defines a λ independent canonical transformation. In terms of new variables the Hamiltonian takes a particularly simple form

$$\hat{H} = \sqrt{2}\omega(\hat{Q}_1\hat{P}_2 - \hat{Q}_2\hat{P}_1) - \frac{m\omega^2}{2}(\hat{Q}_1^2 + \hat{Q}_2^2) \tag{21}$$

Again, it is a sum of two commutating pieces: the first proportional to the angular momentum operator while the second represents the length of $\hat{\vec{Q}}$ squared. Therefore, the energy is the sum of discrete and continuous parts and depends on the combination of two quantum numbers [1].

In what follows we find it convenient to use the momentum representation: $\hat{P}_i \to P_i$, $\hat{Q}_i \to i\hbar\frac{\partial}{\partial P_i}$. Let us define the polar coordinates in momentum space by $P_1 = P\cos\Theta$, $P_2 = P\sin\Theta$. Then

$$\hat{H} = -i\sqrt{2}\omega\hbar\frac{\partial}{\partial\Theta} + \frac{m\omega^2\hbar^2}{2}\left(\frac{\partial^2}{\partial P^2} + \frac{1}{P}\frac{\partial}{\partial P} + \frac{1}{P^2}\frac{\partial^2}{\partial\Theta^2}\right); \tag{22}$$

the spectrum and normalized eigenfunctions read

$$E_{n,k} = \omega\hbar\left(\sqrt{2}n - \frac{m\omega\hbar k^2}{2}\right) \tag{23}$$

$$\Psi_{n,k}(\vec{P}) = \sqrt{\frac{k}{2\pi}}J_n(kP)e^{in\Theta}$$

We shall now consider the limit $\lambda \to \frac{1}{4\omega^2}$. This is slightly subtle due to the fact that the spectrum of the Hamiltonian changes in this limit from discrete into continuous one. Let us put

$$1 - 4\lambda\omega^2 \equiv \varepsilon^2, \quad \varepsilon \to 0^+ \tag{24}$$

Then

$$\omega_{1,2} \simeq \sqrt{2}\omega\left(1 \pm \frac{\varepsilon}{2}\right) \tag{25}$$

The energy spectrum (9) can be rewritten as

$$E_{n_1,n_2} = \sqrt{2}\omega\hbar(n_2 - n_1) - \frac{\sqrt{2}}{2}\omega\hbar\varepsilon(n_1 + n_2 + 1) \tag{26}$$

In the limit $\varepsilon \to 0^+$ the energy seems to be given by the formula [11]

$$E_{n_1,n_2} = \sqrt{2}\omega\hbar(n_2 - n_1) \tag{27}$$

This is, however, not the case. We should take into account that the limiting Hamiltonian has a continuous spectrum given by eq.(24). Therefore, the proper way of taking the limit is to let $n_1, n_2 \to \infty$ in such a way that

$$\begin{aligned} n &= n_2 - n_1 \\ \varepsilon(n_1 + n_2) &= \frac{m\omega\hbar k^2}{\sqrt{2}} \end{aligned} \tag{28}$$

are fixed.
We shall show that, indeed, by considering this limiting procedure one recovers the wave functions (23) of degenerate Hamiltonian.

Let us note that we cannot take the equal frequency limit directly. This is due to the fact that the very coordinate representation becomes singular in this limit as is clearly seen from eq.(7). On the other hand, the momentum representation based on $\hat{P}_1, \hat{P}_2$ (cf. eqs.(22), (23)) is always well-defined. Therefore, the first step will be to rewrite our wavefunctions (10) in momentum representation. To this end we write $\hat{P}_1$ and $\hat{P}_2$ in terms of $\hat{x}_i's$ and $\hat{p}_i's$. By virtue of eqs.(7) and (20) we find

$$\begin{aligned}
\hat{P}_1 &= \frac{b_2\hat{p}_1 + b_1\hat{p}_2}{m\sqrt{\lambda(\omega_1^2-\omega_2^2)}} = \frac{-i\hbar}{m\sqrt{\lambda(\omega_1^2-\omega_2^2)}}\left(b_2\frac{\partial}{\partial x_1} + b_1\frac{\partial}{\partial x_2}\right) \\
\hat{P}_2 &= \frac{\sqrt{2}\omega(-b_1\hat{x}_1 + b_2\hat{x}_2)}{\sqrt{\lambda(\omega_1^2-\omega_2^2)}} = \frac{\sqrt{2}\omega}{\sqrt{\lambda(\omega_1^2-\omega_2^2)}}(-b_1x_1 + b_2x_2) \\
b_{1,2} &\equiv \frac{m}{2\sqrt{2}}\left(\frac{3}{2} + \lambda\omega_{1,2}^2\right)
\end{aligned} \tag{29}$$

It is now straightforward to find the relevant transition functions $\langle x_1, x_2 \mid P_1, P_2\rangle$ by solving the corresponding eigenvalue equations

$$\hat{P}_i\langle x_1, x_2 \mid P_1, P_2\rangle = P_i\langle x_1, x_2 \mid P_1, P_2\rangle \tag{30}$$

Using eqs.(29) one obtains

$$\begin{aligned}
&\langle x_1, x_2 \mid P_1, P_2\rangle = \qquad (31)\\
&= \sqrt{\frac{m\lambda(\omega_1^2-\omega_2^2)}{2\sqrt{2}\pi\hbar\omega}}\,\delta\left(-b_1x_1 + b_2x_2 - \frac{\sqrt{\lambda(\omega_1^2-\omega_2^2)}P_2}{\sqrt{2}\omega}\right) e^{\frac{im\sqrt{\lambda(\omega_1^2-\omega_2^2)}}{\hbar(b_1^2+b_2^2)}P_1(b_2x_1+b_1x_2)}
\end{aligned}$$

This allows us to pass to the momentum representation

$$\langle P_1, P_2 \mid n_1, n_2\rangle = \int dx_1dx_2\langle P_1, P_2 \mid x_1, x_2\rangle\langle x_1, x_2 \mid n_1, n_2\rangle \tag{32}$$

Doing one integration with the help of delta function one arrives at the following rather complicated expression:

$$\begin{aligned}
&\langle P_1, P_2 \mid n_1, n_2\rangle = \frac{N(n_1)N(n_2)}{b_1^2+b_2^2}\sqrt[4]{\frac{m^2\omega_1\omega_2}{\hbar^2}}\sqrt{\frac{m\lambda(\omega_1^2-\omega_2^2)}{2\sqrt{2}\pi\hbar\omega}}\cdot \\
&\cdot e^{\frac{m}{2\hbar}\left[\frac{\lambda(\omega_1^2-\omega_2^2)}{\omega_1b_2^2+\omega_2b_1^2}\left(\frac{b_1b_2(\omega_2-\omega_1)}{\sqrt{2}\omega(b_1^2+b_2^2)}P_2+iP_1\right)^2-\frac{\lambda(\omega_1^2-\omega_2^2)}{2\omega^2}\frac{(\omega_1b_1^2+\omega_2b_2^2)}{(b_1^2+b_2^2)^2}P_2^2\right]}\cdot \qquad (33)\\
&\cdot\int_{-\infty}^{+\infty}dvH_{n_1}\left(\sqrt{\frac{m\omega_1}{\hbar}}\left(\frac{b_2v-b_1\sqrt{\frac{\lambda(\omega_1^2-\omega_2^2)}{\sqrt{2}\omega}}P_2}{b_1^2+b_2^2}\right)\right)H_{n_2}\left(\sqrt{\frac{m\omega_2}{\hbar}}\left(\frac{b_1v+\frac{\sqrt{\lambda(\omega_1^2-\omega_2^2)}}{\sqrt{2}\omega}P_2}{b_1^2+b_2^2}\right)\right)\cdot \\
&\cdot e^{\frac{-m}{2\hbar}\left[\frac{\sqrt{\omega_1b_2^2+\omega_2b_1^2}}{b_1^2+b_2^2}v+\sqrt{\frac{\lambda(\omega_1^2-\omega_2^2)}{\omega_1b_2^2+\omega_2b_1^2}}\left(\frac{b_1b_2(\omega_2-\omega_1)}{\sqrt{2}\omega(b_1^2+b_2^2)}P_2+iP_1\right)\right]^2}
\end{aligned}$$

In principle, the last integral could be also taken. However, this is not necessary. We can take the limit $\varepsilon \to 0^+$ directly in eq. (33). In this limit

$$b_{1,2} \simeq \frac{m}{\sqrt{2}}\left(1 \pm \frac{\varepsilon}{4}\right) \tag{34}$$

Keeping dominant terms in eq.(33) one finds

$$\langle P_1, P_2 \mid n_1, n_2\rangle \simeq \frac{N(n_1)N(n_2)\sqrt{\varepsilon}}{\sqrt{\pi}\hbar} \cdot \tag{35}$$
$$\cdot \int\limits_{-\infty}^{+\infty} dy H_{n_1}\left(y - \sqrt{\frac{\varepsilon}{2\sqrt{2}m\hbar\omega}}(P_2 + P_1)\right) H_{n_2}\left(y - \sqrt{\frac{\varepsilon}{2\sqrt{2}m\hbar\omega}}(-P_2 + iP_1)\right) e^{-y^2}$$

The last integral is taken explicitly yielding [12]

$$\langle P_1, P_2 \mid n_1, n_2\rangle \simeq \tag{36}$$
$$\simeq \frac{\sqrt{\varepsilon}}{\sqrt{\sqrt{2}m\hbar\omega}}\left(\frac{\varepsilon}{2\sqrt{2}m\hbar\omega}\right)^{\frac{n}{2}} (P_2 - iP_1)^n n_1! 2^{n_2} N(n_1)N(n_2) L^n_{n_1}\left(\frac{\varepsilon(P_1^2 + P_2^2)}{\sqrt{2}m\omega\hbar}\right)$$

(for definiteness we have assumed here $n_1 \geq n_2$; the opposite case goes along the same way). Now, it is easy to take the limit $\varepsilon \to 0, \quad n_{1,2} \to \infty$ with $n \equiv n_2 - n_1$ and $\varepsilon(n_1 + n_2) = \frac{m\omega\hbar k^2}{\sqrt{2}}$ fixed. Using Stirling formula and the asymptotic form of Laguerre polynomials [12] we arrive finally at the following result

$$\langle P_1, P_2 \mid n_1, n_2\rangle_{n_1,n_2\to\infty\varepsilon\to 0} \simeq (-i)^n \sqrt{\frac{\sqrt{2}\varepsilon}{m\hbar\omega k}}\left(\sqrt{\frac{k}{2\pi}} J_n(kP)\right) e^{in\Theta} \tag{37}$$

By comparing eq.(36) and (23) we conclude that by taking the limit of equal frequencies in the way prescribed above we recover the wavefunctions of degenerate Hamiltonian. To complete the arguments let us only note that the additional $\sqrt{\varepsilon}$ factor comes from the fact that the discrete eigenfunctions have unit norm while the norm of limiting one is infinite.

Let us now consider the question whether the double-frequency case can be quantized in the way which yields positive-definite Hamiltonian at the expense of having indefinite metric in the space of states. More precisely, the problem is the following: for differening frequencies we choose the positive-energy quantization scheme; is it possible to perform the equal-frequency limit keeping the energy positive?

In order to answer this question one has to adopt some definition of taking the equal-frequency limit. The most reasonable way is to keep the initial operators $\hat{q}_i, \hat{\Pi}_i$ as given once forever linear operators acting in some linear space of states and varying the Hamiltonian by varying some of its parameters (λ in our case). For $\varepsilon > 0$ one can define, via eqs.(7) and (16), the hermitean conjugation "+" for $q_i's$ and $\Pi_i's$ (and, hence, convert the space of states into usual Hilbert space) in such a way that the Hamiltonian

(5) becomes a positive-definite operator. The definition of hermitean conjugation " + " depends, of course, on ε. It is easy to compute, using (7) and (16), that, for example

$$\hat{q}_1^+ \simeq \frac{1}{\varepsilon^2}\left(\hat{q}_1 - \frac{2}{m}\hat{\Pi}_2\right) \tag{38}$$

Therefore, the reasonable conjugation rule cannot be imposed in the limiting case. One can support this conclusion by considering the operators a and b defined by eqs. (33) of Ref. [8]. In fact, it is easy to check that a diverges if the limit is taken in the way prescribed above; this is not in contradiction with the fact that their commutators and the Hamiltonian (eqs. (35), (40) of Ref [8]) are well-behaving in the limit $\varepsilon \to 0$.

However, one can argue that our prescription of taking the $\varepsilon \to 0$ limit is not the only possible. In fact, one can argue that it is sufficient to get the regular limiting commutation rules and the Hamiltonian. This seems to be the strategy adopted by the authors of Ref. [8].

Let us consider in some detail their construction. The limiting commutation rules and the Hamiltonian read

$$\begin{aligned} &[a, a^\star] = 0, \quad [b, b^\star] = 0, \quad [b, a^\star] = \mu, \quad [a, b^\star] = \mu, \quad [a, b] = 0 \\ &H = \frac{\omega}{\mu}(2b^\star b + a^\star b + b^\star a) + const \end{aligned} \tag{39}$$

By defining

$$A_1 = \frac{1}{\sqrt{2\mu}}\,(a+b)\,, \quad A_2 = \frac{1}{\sqrt{2\mu}}\,(a-b) \tag{40}$$

one obtains

$$\begin{aligned} &[A_i, A_j] = 0, \quad [A_i^\star, A_j^\star] = 0, \quad [A_i, A_j^\star] = 0, i \neq j, \quad [A_1, A_1^\star] = 1, \quad [A_2, A_2^\star] = -1 \\ &H = \omega(2A_1^\star A_1 - A_1^\star A_2 - A_2^\star A_1) \end{aligned} \tag{41}$$

Then we obtain the indefinite Fock space for two degrees of freedom. Define new metric operator

$$\tau \equiv e^{i\pi A_2^+ A_2} \tag{42}$$

and new conjugation

$$B^+ \equiv \tau B^\star \tau \tag{43}$$

Then $A_1^+ = A_1^\star$, $A_2^+ = -A_2^\star$ and one obtains standard commutation rules together with Fock representation with positive metric and " + " playing the role of hermitean conjugation.

Now, H can be viewed as acting in standard Hilbert space,

$$H = \omega(2A_1^+ A_1 - A_1^+ A_2 + A_2^+ A_1) \tag{44}$$

We see that H is not normal

$$[H, H^{+}] \neq 0 \tag{45}$$

so it cannot be diagonalized. Moreover,,

$$[H, N] = 0 \tag{46}$$

where $N \equiv A_1^+ A_1 + A_2^+ A_2$ is the total number operator. Therefore, we can consider H as acting separately in each finitedimensional subspace of fixed eigenvalue of N. It is represented as a matrix in any such subspace and we want to determine its Jordan form. To this end consider the subspace spanned by $\mid n_1, n_2\rangle,\ n_1 + n_2 = n$; then

$$\begin{aligned}
&(H - \omega n)^{n+1} \mid n_1, n_2\rangle = (H - \omega N)^{n+1} \mid n_1, n_2\rangle = \\
&= \omega^{n+1}(A_1^+ A_1 - A_2^+ A_2 - A_1^+ A_2 + A_2^+ A_1)^{n+1} \mid n_1 n_2\rangle = \\
&= \omega^{n+1}((A_1^+ + A_2^+)(A_1 - A_2))^{n+1} \mid n_1, n_2\rangle = \\
&= \omega^{n+1}(A_1^+ + A_2^+)^{n+1}(A_1 - A_2)^{n+1} \mid n_1, n_2\rangle = 0
\end{aligned} \tag{47}$$

because $[A_1^+ + A_2^+,\ A_1 - A_2] = 0$. Therefore, in any subspace corresponding to the eigenvalue n of N H takes the form of single Jordan block; any such subspace contains exactly one eigenvector corresponding to the eigenvalue $n\omega$. In order to construct it we note that

$$[H, A_1^+ + A_2^+] = \omega(A_1^+ + A_2^+) \tag{48}$$

Therefore

$$\mid n\rangle = (A_1^+ + A_2^+)^n \mid 0, 0\rangle \tag{49}$$

is the eigenvector of H corresponding to the eigenvalue $n\omega$ of H. Now, using

$$\mid n\rangle = \sum_{k=0}^{n} \binom{n}{k} \sqrt{k!}\sqrt{(n-k)!} \mid k, n-k\rangle \tag{50}$$

one finds

$$\langle n \mid n\rangle = n! \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} = 0 \tag{51}$$

We see that all eigenvectors of H have zero norm. However, it is reasonable to assume that the physical space of states is spanned by the eigenvectors of H. This is a subspace of zero norm so it can hardly be viewed as physical one.

Concluding, we see no way to get the positive-energy quantized degenerate PU theory, at least by limiting procedure.

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